On the Solution of Two-Point Linear Differential Eigenvalue Problems

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A numerical technique is presented for locating the eigenvalues of two point linear differential eigenvalue problems. The technique is designed to search for complex eigenvalues belonging to complex operators. With this method, any domain of the complex eigenvalue plane could be scanned and the eigenvalues within it, if any, located. For an application of the method, the eigenvalues of the Orr–Sommerfeld equation of the plane Poiseuille flow are determined within a specified portion of the c-plane. The eigenvalues for $\alpha = 1$ and $R = 10,000$ are tabulated and compared for accuracy with existing solutions.

INTRODUCTION

In this paper a numerical method is outlined for locating the spectrum of complex eigenvalues belonging to a complex linear differential operator in a prescribed section of the complex eigenvalue plane. Full use is made here of the fundamental properties of the solution of linear differential operators, and hence, the shooting technique is employed in the numerical procedure. This method is presented in contrast to the finite difference techniques which were used almost exclusively in the past for locating the eigenvalues. In the finite difference procedure, the original differential eigenvalue problem is transformed to an algebraic one, and thus, the eigenvalues obtained are those belonging to the algebraic problem. However, the finite difference technique possesses two shortcomings. First, the number of eigenvalues sought and determined is directly proportional to the number of mesh points used in the differencing algorithm. Secondly, there does not exist an a priori control on the section of the complex eigenvalue plane to be searched. The present technique does allow the user to search any portion of the eigenvalue plane that he may wish, and certainly the number of eigenvalues to be determined in no way depends on the integration method. Also, it is worth mentioning here that this technique may be used for the location of the real spectrum of eigenvalues belonging to a real operator.

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Complex eigenvalues belonging to complex differential operators arise in the theory of hydrodynamic stability; therefore, the problem of the stability of the plane Poiseuille flow to infinitesimal disturbances is discussed and the pertinent results are presented as an illustration of the method. Most of the accurate techniques for solving such an eigenvalue problem are called *local* methods, in which an a priori knowledge of the location of the eigenvalues is necessary. Since local methods are iterative techniques, the convergence of the process and its speed depends on how well an initial guess is made, so that one is forced to come up with a good initial guess. In the past, information for constructing a good initial guess has been obtained either from known existing solutions or generated by techniques such as the variational method of Lee and Reynolds [1]. It is believed that the present technique may also provide good estimates for the location of eigenvalues in the desired domain of the eigenvalue plane for use in *local* methods

Controversy still remains relative to the nature of the spectrum of eigenvalues of the linear stability problem of many flows profiles. The discussion arises as to whether the spectrum is infinite or finite for some flows and whether there exist any unstable modes for other types of flows. The present method could be used to answer some of these questions, if not in the whole of the complex plane, then at least for a certain finite portion of the plane. The theoretical background necessary for the development of the method is presented in the next section. The application of the method to the plane Poiseuille flow is illustrated in the section after next, and a general discussion of the results is given in the last section.

THEORETICAL BACKGROUND

Most linearized problems of mathematical physics are usually reduced to eigenvalue problems for their solution. Such problems are slightly more difficult to solve than regular boundary value problems because in seeking solutions of such problems, the eigenvalues as well as the eigenfunctions must be determined. In some cases a knowledge of the eigenvalues alone provides a good deal of insight into the problem and so merits their determination. When the coefficients of the differential operator are variables, most often a numerical procedure is the only recourse to its solution. In what follows we present such a procedure.

Consider the eigenvalue problem

$$L_1(u) + \lambda L_2(u) = 0, \tag{1a}$$

where $L_1(u)$ and $L_2(u)$ are linear differential operators of the form

$$L(u) = p_0(x) D^n u + p_1(x) D^{n-1} u + \cdots + p_n(x)u.$$

λ in (1a) is a complex eigenvalue and u is a complex function of the real variable x. Also in the above, the coefficients $p_k(x)$ possess continuous derivatives up to order n on the interval $[a, b]$ and D denotes differentiation with respect to x. Let the differential equation (1a) be subject to the following boundary conditions:

$$U_i(u) = 0, \qquad (i = 1,..., n) \tag{1b}$$

which are specified on both ends of the interval $[a, b]$. In the above, $U_i(u)$ are the homogeneous linear forms in the $2n$ quantities:

$$u(a), \qquad Du(a),..., D^{n-1}u(a),$$
$$u(b), \qquad Du(b),..., D^{n-1}u(b).$$

Let us define the determinant $\Delta(\lambda)$ by:

$$\Delta(\lambda) = \det \begin{bmatrix} U_1(u_1) & \cdots & U_1(u_n) \\ \vdots & & \vdots \\ U_n(u_1) & \cdots & U_n(u_n) \end{bmatrix}, \tag{2}$$

where $u_1, u_2,..., u_n$ is a fundamental system of Cauchy solutions of (1a), collectively satisfying linearly independent sets of initial conditions at $x = a$. For the case where the coefficients of the differential operators, namely $p_n(x)$, and the boundary conditions $U_i(u)$ are independent of the eigenvalue λ, the following theorem can be invoked (see Naimark [2]): *The eigenvalues of the operator equation* (1a) *together with the boundary conditions* (1b) *are the zeroes of the function* $\Delta(\lambda)$. *If* $\Delta(\lambda)$ *vanishes identically, then any number* λ *is an eigenvalue of problem* (1). *If, however,* $\Delta(\lambda)$ *is not identically zero, problem* (1) *has at most denumerably many eigenvalues, and the eigenvalues have no finite limit point.*

It can also be shown that if the operators L_1 and L_2 are not singular, then $\Delta(\lambda)$ is an integral analytic function of λ.

By using the above theorem, it is evident that to determine the eigenvalues of problem (1), one only needs to locate the zeroes of the function $\Delta(\lambda)$. In this work the interest lies in locating the eigenvalues of problem (1) within a finite region of the λ plane; therefore, in what follows we will consider only the zeroes of $\Delta(\lambda)$ within a bounded region of the λ plane. Since $\Delta(\lambda)$ is a complex function of the complex number λ, the zeroes of $\Delta(\lambda)$ can be determined with the aid of the following theorem (see Goodstein [3]): *If* γ *is a regular curve in a simply connected open set not passing through any zero or pole of a meromorphic function* $\Delta(\lambda)$ *in the set, the integral*

$$\frac{1}{2\pi i} \int_\gamma \frac{\Delta'(\lambda)}{\Delta(\lambda)} \, d\lambda, \tag{3}$$

is equal to the excess of the number of zeroes over the number of poles of $\Delta(\lambda)$ inside γ (a pole or zero of order k counting as k poles or zeroes, respectively). The prime in (3) denotes differentiation with respect to λ.

As pointed out earlier, $\Delta(\lambda)$ is an analytic function of λ; therefore, within any closed region in the λ plane there are no poles and, hence, (3) could be written as

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\Delta'(\lambda)}{\Delta(\lambda)} \, d\lambda = M, \tag{4}$$

where M is the number of zeroes within the curve γ.

The above theorem admits a simple generalization. As before, let $\Delta(\lambda)$ be analytic inside and on γ and let $\omega(\lambda)$ be also analytic inside and on γ; then

$$\frac{1}{2\pi i} \int_{\gamma} \omega(\lambda) \frac{\Delta'(\lambda)}{\Delta(\lambda)} \, d\lambda = \sum_{r=1}^{M} q_r \omega(a_r), \tag{5}$$

where the sum is taken over the zeroes of $\Delta(\lambda)$ and q_r is the order of the zeroes, a_r. Expression (5) will be used in the next section for locating the position of the eigenvalues inside the curve γ.

The Orr–Sommerfeld Equation

Formulation of the Problem

To clarify the method outlined in the previous section and to facilitate an example, the eigenvalue spectrum of the Orr-Sommerfeld equation is investigated here. This equation describes the stability of two-dimensional, incompressible parallel flows with regard to infinitesimal disturbances. It takes the following form (see Lin [4]):

$$(D^2 - \alpha^2)^2 \, \phi = i\alpha R\{(u - c)(D^2 - \alpha^2)\phi - \phi D^2 u\}, \tag{6}$$

where D denotes differentiation with respect to y, α is the disturbance wave number, c its speed, and R is the flow Reynolds number. In Eq. (6), u is the undisturbed laminar profile, which for the plane Poiseuille flow is a quadratic function of y; i.e., $u = 1 - y^2$. The usual boundary conditions for Eq. (6) are

$$\phi(\pm 1) = 0 = D\phi(\pm 1). \tag{7}$$

However, Eq. (6) is symmetrical in y, hence it is sufficient to consider only odd or even solutions to the eigenvalue problem comprised of (6) and (7). Thus,

Eq. (6) could be solved for half the range, [0, 1], by replacing the conditions given in (7) with the following ones:

$$D\phi(0) = D^3\phi(0) = 0 \quad \text{for even solution}$$
$$\phi(0) = D^2\phi(0) = 0 \quad \text{for odd solution} \tag{8}$$
$$\phi(1) = D\phi(1) = 0$$

From the two choices given in (8), the boundary condition appropriate to the even solution is the one chosen for the discussion below because the even solution is the only one containing an unstable mode.

Rewriting Eq. (6) in operator form yields

$$L_1(\phi) + cL_2(\phi) = 0, \tag{9}$$

where

$$L_1 = \{(D^2 - \alpha^2)^2 - i\alpha R[u(D^2 - \alpha^2) - D^2u]\},$$
$$L_2 = i\alpha R(D^2 - \alpha^2). \tag{10}$$

The eigenvalue here is the wave speed c, which is a complex number; i.e., $c = c_r + ic_i$ (from this point onward c will replace λ of the previous section). The boundary conditions could also be rewritten in the form of the previous section; i.e.,

$$U_1 = D\phi(0) = 0,$$
$$U_2 = D^3\phi(0) = 0,$$
$$U_3 = \phi(1) = 0, \tag{11}$$
$$U_4 = D\phi(1) = 0.$$

Based on these boundary conditions, the characteristic determinant $\Delta(\lambda)$ defined in (2) will then take the form

$$f(c) = \det \begin{bmatrix} D\phi_1(0) & D\phi_2(0) & D\phi_3(0) & D\phi_4(0) \\ D^3\phi_1(0) & D^3\phi_2(0) & D^3\phi_3(0) & D^3\phi_4(0) \\ \phi_1(1) & \phi_2(1) & \phi_3(1) & \phi_4(1) \\ D\phi_1(1) & D\phi_2(1) & D\phi_3(1) & D\phi_4(1) \end{bmatrix}. \tag{12}$$

Since the boundary conditions, (10), are separable with two of the conditions applying at $y = 1$ and the remaining two at $y = 0$, the order of the characteristic determinant, (12), can be reduced with great saving of labor by solving an initial value problem defined by augmenting the given conditions at $y = 0$ by two further sets of two linearly independent conditions at that point. When the resulting

fundamental solutions are $\bar{\phi}_1$ and $\bar{\phi}_2$ (where $\bar{\phi}_i = \{\phi_i, D\phi_i, D^2\phi_i, D^3\phi_i\}$), (12) is replaced by the characteristic determinantal equation expressing the compatibility of a nontrivial combination of these solutions and their derivatives with the given boundary conditions at $y = 1$; namely

$$f(c) = \det \begin{bmatrix} \phi_1(1) & \phi_2(1) \\ D\phi_1(1) & D\phi_2(1) \end{bmatrix} \tag{13}$$

Numerical Details

To locate the zeroes of $f(c)$ inside a prescribed domain in the c-plane, a numerical method developed by Delves and Lyness [5] was used. The method is only sketched here, and for details the reader is advised to consult the reference. First, the number of zeroes inside the specified region is determined through an approximation to the contour integral, (4), by a suitable quadrature formula (the formula suggested in [5] was used here). In the present investigation the contour of integration, γ, was taken to be a circle following Delves and Lyness [5]. Once the number of zeroes inside γ, say M, was determined, then their location was found in the second step. Letting the function $\omega(\lambda)$ in (5) take successively the values $c, c^2, c^3, ..., c^M$, the various power sums of the zeroes were determined by performing the following M contour integrations:

$$s_1 = \sum_{j=1}^{M} c_j = \int_{\gamma} c \frac{f'(c)}{f(c)} dc,$$

$$s_2 = \sum_{j=1}^{M} c_j^2 = \int_{\gamma} c^2 \frac{f'(c)}{f(c)} dc, \tag{14}$$

$$\vdots$$

$$s_M = \sum_{j=1}^{M} c_j^M = \int_{\gamma} c^M \frac{f'(c)}{f(c)} dc,$$

where c_j are all the zeroes of $f(c)$ inside γ. Again, the values of the contour integrals in (14) were obtained through the use of a suitable quadrature formula.

Using the numbers $s_1, s_2, ..., s_M$, a polynomial $P(c)$ of degree M could be constructed whose zeroes coincide with the zeroes of $f(c)$. For the construction of such a polynomial, Newton's formula is used in which the coefficients of the polynomial $P(c)$ are evaluated from the power sums of the roots of the polynomial. Once the equivalent polynomial $P(c)$ is constructed, then any polynomial root finder could be employed to extract the zeroes of $P(c)$. The zeroes of $f(c)$ are coincident with the roots of $P(c)$. The polynomial root finder subroutine used in the present investigation utilizes the quadratic method. It should be mentioned

here that the numerical technique for finding the zeroes of a function described above is capable of determining multiple zeroes of the function if any exist. However, multiple zeroes were not found in the present problem within the domain of search.

Examination of the quadrature expressions in (4) and (5) reveals that the evaluation of the derivative with respect to c of the secular determinant is necessary. This derivative is obtained by the usual rule of determinant differentiation which requires the derivative of each element of (13) with respect to c evaluated at $y = 1$. To achieve this, two more initial value problems are integrated from 0 to 1 whose governing equations are

$$L_1 \left(\frac{\partial \phi_i}{\partial c} \right) + c L_2 \left(\frac{\partial \phi_i}{\partial c} \right) + L_2(\phi_i) = 0 \qquad (i = 1, 2) \tag{15}$$

In Eq. (15) L_1 and L_2 are the differential operators defined in (10). Note that since f is an analytic function of c, then the derivative is taken with respect to either c_r or c_i. The initial value problems defined in Eq. (15) are then solved subject to the following homogeneous initial conditions:

$$\frac{\partial}{\partial c} (\phi_i(0)) = \frac{\partial}{\partial c} (D\phi_i(0)) = 0 = \frac{\partial}{\partial c} (D^2\phi_i(0)) = \frac{\partial}{\partial c} (D^3\phi_i(0)) \qquad (i = 1, 2). \tag{16}$$

Hence, for the complete solution of the problem, the linearly independent solutions $\bar{\phi}_1$ and $\bar{\phi}_2$ are augmented by two further solutions for $\partial \bar{\phi}_1/\partial c$ and $\partial \bar{\phi}_2/\partial c$.

The Orr–Sommerfeld equation (6) presents a problem when it is integrated numerically. The difficulty is due to the fact that some of the coefficients are widely separated. This is borne out when it is observed that the wave number, α, is of order one while the Reynolds number R is of order 10^3 or more. Since in the numerical integration of this equation one starts with linearly independent (orthogonal) initial vectors, because of the wide separation of the coefficients these vectors become increasingly parallel as the integration proceeds to the final point, resulting in their being linearly dependent.

Many methods have been devised to overcome this situation, and, to the author's knowledge, they are all implementations of one basic idea. Since the initial vectors were orthogonal, it is necessary to ensure that these vectors remain orthogonal throughout the range of integration. This is achieved by reorthonormalizing the solution vectors at discrete points along the path of integration. Conte's [6] method is followed in this work which consists of using the Gramm–Schmidt orthonormalization technique, and for details of the method the reader is advised to consult that reference. In the present work, however, it is also neces-

sary to orthonormalize the solution vectors for $\partial \bar{\phi}_1/\partial c$ and $\partial \bar{\phi}_2/\partial c$; and for that end the same orthonormalization transformation is used as the one for the solution vectors of $\bar{\phi}_1$ and $\bar{\phi}_2$.

RESULTS AND DISCUSSION

Since the Orr–Sommerfeld equation for the plane Poiseuille flow has received much attention by workers in both analytical and numerical methods, it is considered a good example for discussing the merits of the present technique. Only recently has any attempt been undertaken to tabulate the eigenvalue spectrum of Eq. (6). Specifically, there exist two such tabulations in the open literature. The first is by Orszag [7], who tabulated 17 symmetrical and 15 antisymmetrical eigenmodes of Eq. (6) for $\alpha = 1$ and $R = 10,000$. His method, however, is not designed to search a specific portion of the eigenvalue plane, but the eigenmodes obtained are those of a transformed algebraic eigenvalue problem. The second is by Mack [8]. Employing a different search scheme from the present one, he tabulated the 32 known symmetric eigenmodes (for the same problem as that of Orszag's) that lie in the rectangle $0 \leqslant c_r \leqslant 1.0$, $-1.1 \leqslant c_i \leqslant 0$, of the complex c-plane (the first 17 eigenvalues are those computed by Orszag). The results of the present scheme are compared with the latter tabulation.

As was shown earlier, for each point evaluation of the contour integral, two solutions of Eq. (6) and two of Eq. (15) are required. Since both of these equations are fourth order, they are solved as a system of four first-order equations; therefore, in all a total of 16 complex or 32 real equations are integrated forward over the interval [0, 1]. The initial value integrators commonly used in such cases are of the Runge-Kutta type with either a fixed or variable step size. However, both Eq. (6) and Eq. (15) are special in that they could be reduced to a system of two second-order equations which do not contain the first derivative explicitly. Runge-Kutta–Nyström formulas, developed by Fehlberg [9], for integrating such systems of second-order differential equations were used in the present analysis. Thus, instead of integrating 32 real equations, only 16 real equations are integrated, resulting in a great saving of the computation time. The specific formulas used here are those of a seventh-order Runge–Kutta–Nyström with a fixed step size of 0.05.

The search procedure employed in this example is the following. The region desired to be searched (for the present problem the rectangle $0 \leqslant c_r \leqslant 1.0$ and $-1.1 \leqslant c_i \leqslant 0$) was spanned by circles of radius 0.071 as shown in Fig. 1. Then using the technique outlined in the previous section, each circle was searched in turn for eigenvalues. Although it is desirable to span the whole domain of interest by one circle, it was found that there exists an upper limit on the size of the search
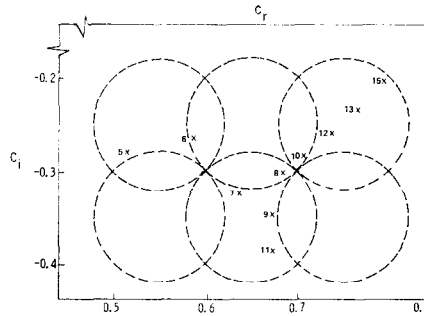
FIG. 1. A section of the c-plane showing the method of search for the eigenvalues. This section is spanned by six circles containing modes 5–13 and 15.

contour. It is observed that whenever the radius of the circle in question was larger than a specific value, the quadrature formula failed to converge (for the present problem this value was 0.1). This failure is primarily caused by the problem of separation of coefficients of Eq. (6) and not by any property of the quadrature formulas themselves.

All of the eigenvalues found in this region are indicated in Fig. 2 and listed in Table I. (The eigenvalues are numbered in ascending order according to their magnitude in the lower half plane.) In Table I, $| c |$ is the magnitude of the eigenvalue; c_r and c_i are its real and imaginary parts, while m is the number of function evaluations used in the quadrature formulas. In the sixth column, the percentage
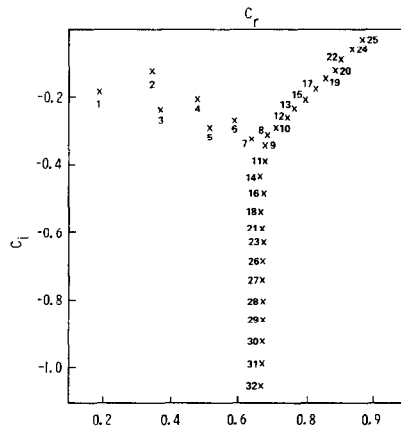

FIG. 2. The rectangle in the lower half of the c-plane which is searched for the Orr–Sommerfeld problem of the plane Poiseuille flow for $\alpha = 1.0$ and $R = 10,000$. All the stable modes for the even solution are shown in the rectangle $0 \leqslant c_r \leqslant 1.0$; $-1.1 \leqslant c_i \leqslant 0$.

TABLE I^a

Mode	$\lvert c \rvert$	c_r	c_i	m	Error^b
01	0.26372	0.19***	−0.182**	512	2.1 E-03
02	0.37065	0.34***	−0.1245*	64	1.0 E-03
03	0.43912	0.3685*	−0.2388*	32	3.4 E-05
04	0.51875	0.47490	−0.20873	64	0.0
05	0.58757	0.51292	−0.28663	128	0.0
06	0.64513	0.587**	−0.26716	128	1.3 E-04
07	0.71440	0.63610	−0.32519	128	0.0
08	0.74895	0.6828*	−0.30761	128	8.2 E-06
09	0.75979	0.677**	−0.34373	128	5.7 E-05
10	0.76502	0.70887	−0.2876*	256	9.1 E-06
11	0.77906	0.67541	−0.38983	128	0.0
12	0.78246	0.73812	−0.2596*	256	1.3 E-05
13	0.80071	0.76649	−0.23159	256	0.0
14	0.80196	0.6****	−0.43***	128	9.1 E-03
15	0.82047	0.79482	−0.20353	256	0.0
16	0.82798	0.67232	−0.48326	128	0.0
17	0.84164	0.82314	−0.17548	128	0.0
18	0.85703	0.67159	−0.53241	128	0.0
19	0.86412	0.85145	−0.14740	256	0.0
20	0.88782	0.87976	−0.11937	256	0.0
21	0.88905	0.67097	−0.58327	128	0.0
22	0.91264	0.90***	−0.09***	128	9.3 E-04
23	0.92402	0.67043	−0.63588	128	0.0
24	0.98348	0.93***	−0.06***	128	2.0 E-03
25	0.96193	0.66997	−0.69025	128	0.0
26	0.96528	0.96***	−0.03***	128	1.0 E-03
27	1.0027	0.6****	−0.74***	128	4.2 E-04
28	1.0464	0.66923	−0.80439	128	0.0
29	1.0928	0.66894	−0.86418	128	0.0
30	1.1420	0.66868	−0.92582	64	0.0
31	1.1940	0.66846	−0.98932	64	0.0
32	1.2486	0.66826	−1.05468	256	0.0

^{a} Eigenvalues of Eq. 6 with $\alpha = 1.0$ and $R = 10^4$.
^{b} Error $= \lvert c_{\text{ref. [8]}} - c \rvert / \lvert c_{\text{ref. [8]}} \rvert$.

error between Mack's and the present calculation is indicated. The convergence criterion for the evaluations of Eqs. (4) and (5) was that whenever the difference between successive approximations of the quadrature values was less than 1 percent, these values were considered converged. As is seen in Table I, this criterion is good enough to give up to 5-digit accuracy for many of the eigenvalues. The asterisk in some of these numbers indicates the specific digit that did not agree with the known accurate values (Mack has tabulated the eigenvalues up to 5 digits only). This apparent discrepancy is not as bad as it seems because, although the digits differed, the magnitude of the eigenvalues, c, was less than 1 percent of the known accurate values.

All of the computations were performed on the IBM 360/65 computer in double precision arithmetic whenever possible. Since the time required for each function evaluation is considerable (for each such evaluation the integration of 16 differential equations over the interval [0, 1] was necessary), the computation time for each eigenvalue determination is directly proportional to the value of m. As is shown in Table I, for the majority of eigenvalues, 128 or less function evaluations were necessary. The time required for 128 such evaluations was about 3 minutes. Some difficulties were encountered, however, in the vicinity of the origin of c-plane, resulting in poor convergence; e.g., 512 function evaluations were necessary for mode 1. This difficulty is, again, due to the wide separation of the coefficients of Eq. (6) which causes considerable growth of the solution of the differential equations even with many orthonormalizations.

A word of caution is in order here; although the method works quite well in locating the eigenvalues, the eigenfunctions are not determined. Hence, a local method together with this method is necessary for both the determination of the eigenvalues to a much higher accuracy (greater than 5 digits) and the determination of the eigenfunctions. It is also thought that the requirement of high accuracy from the present technique will be very taxing in computation time. This requirement is unnecessary since local methods will converge much faster once the approximate locations of the eigenvalues are known.

The following conclusions can be drawn from the above discussion concerning the method under investigation:

(i) The present scheme could be used in searching a desired portion of the complex eigenvalue plane to locate the eigenvalues for a linear differential operator. The method is deterministic and does not require any given or a priori knowledge of the location of these eigenvalues. Furthermore, if high accuracy on the eigenvalues is not desired, the present method is sufficient.

(ii) When high accuracy on the eigenvalues is required, the present scheme can be used to search for the eigenvalues; and, once located, these values could be inputted into a local method which will converge fast enough to the desired

accuracy on the eigenvalues and eigenfunctions. Thus, this method could save considerable time by eliminating the trial and error and guesswork from the local methods.

REFERENCES

1. L. H. LEE AND W. C. REYNOLDS, *Q. J. Appl. Math.* **20** (1967), 1.
2. M. A. NAIMARK, "Linear Differential Operators," Part I, Chap. 1, Frederick Ungar, New York, 1967.
3. R. L. GOODSTEIN, "Complex Functions," Chap. 8, McGraw–Hill, New York, 1965.
4. C. C. LIN, "The Theory of Hydrodynamic Stability," Chap. 3, Cambridge Univ. Press, Cambridge, 1967.
5. L. M. DELVES AND J. N. LYNESS, *Math. of Comp.* **21** (1967), 543.
6. S. D. CONTE, *SIAM Rev.* **8** (1966), 309.
7. S. A. ORSZAG, *J. Fluid Mech.* **50** (1971), 689.
8. L. M. MACK, *J. Fluid Mech.* (1976), in press.
9. E. FEHLBERG, *Computing* **10** (1972), 305.